A wave geometry diagnostic

Harnik, N., and R. S. Lindzen, 2001: The effect of reflecting surfaces on the vertical structure and variability of stratospheric planetary waves. *JAS* **58**, 2872-2894.

A diagnostic of the basic-state wave propagation characteristics, which is particularly useful for determining the existence and location of turning surfaces for meridional and vertical propagation. The diagnostic used is a more accurate indicator of wave propagation regions than the index of refraction because it diagnoses meridional and vertical propagation separately. I am happy to share the fortran codes for this diagnostic or assist in writing you own, upon request.

First some background classical wave theory. In Cartesian coordinates, the Rossby wave equation written in terms of the geopotential stream function is:

1) $$\frac{\partial^2\psi}{\partial z^2}+\frac{N^2}{f^2}\frac{\partial^2\psi}{\partial y^2}+n_{ref}^2\psi=0$$

Where n^2_{ref} is the index of refraction squared, which equals:

2) $$n_{ref}^2=\frac{N^2}{f^2}\left[\frac{\overline{q}_y}{U-c}-k^2\right]-\frac{1}{4H^2}$$

Where *U, q_y*, and N^2 are the zonal mean wind, meridional PV gradient and Brunt Vaisala frequency, *H* the density scale height, *f* the Coriolis parameter, and *k, c* are the zonal wavenumber and phase speeds, respectively. For stationary waves, *c=0.*

When n^2_{ref} is separable in the latitude and height directions, a wave equation can be written separately in each direction, and a solution can be obtained which his either a propagating wave or an evanescent perturbation:

3) $\frac{\partial^2\psi}{\partial z^2}+m^2\psi=0$; where the solution depends on the sign of m^2 (in WKB form):

- $m^2<0$, $\psi=\frac{A}{\sqrt{m}}e^{i\int mdz}+\frac{B}{\sqrt{m}}e^{-i\int mdz}$ - **wave propagation**.

  The solution is a superposition of upward (positive exponent) and downward (negative exponent) propagating waves

- $m^2>0$, $\psi=\frac{A}{\sqrt{m}}e^{\int mdz}+\frac{B}{\sqrt{m}}e^{-\int mdz}$ - wave evanescence.

The solution is a superposition of an exponentially growing and an exponentially decaying components. For an open domain only the negative exponent satisfied the top boundary condition

- m^2=0, a **turning surface.** Waves propagating to such a surface get reflected

- $m^2 \rightarrow \pm\infty$, Such a surface occurs where the waves move with the background flow (U=c) and is called a **critical surface.** This is where waves interact with the mean flow, and get absorbed or overreflected.

For typical mean flows, n^2_{ref} is not separable in the latitude and height directions so that the division to vertical and meridional wave propagation is not trivial. In this case, Harnik and Lindzen (2001) showed that this separation can be diagnosed from the steady state solution to (1) as follows (where m^2 and l^2 are the vertical and meridional parts of n^2_{ref}, respectively):

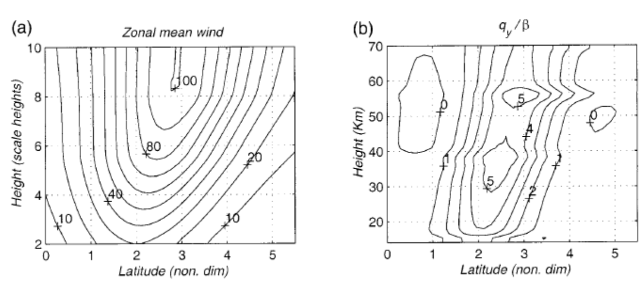
4) $$m^2 \equiv -\mathrm{Re}\left(\frac{\psi_{zz}}{\psi}\right)$$

5) $$l^2 \equiv -\mathrm{Re}\left(\frac{\psi_{yy}}{\psi}\right)$$

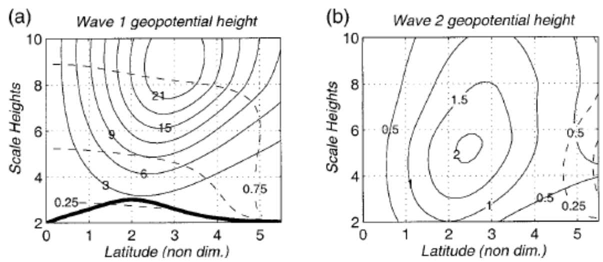
Thus, for a given zonal mean flow structure, for specified zonal wavenumber *k* and zonal phase speed *c*, we can calculate the vertical and meridional wavenumbers *m* and *l*, which if real are indicative of wave propagation and if imaginary are indicative of wave evanescence. The lines of zero *m* and *l* are the correspondingly the reflecting surfaces for vertical or meridional propagation. Thus, *m* and *l* are diagnostics **of the mean flow propagation characteristics for the particular zonal mode *k*, *c*, and not of the wave itself**, which can vary in time, depending strongly on the characteristics and evolution of the wave sources and sinks.

Example from a linear QG model on a β plane:

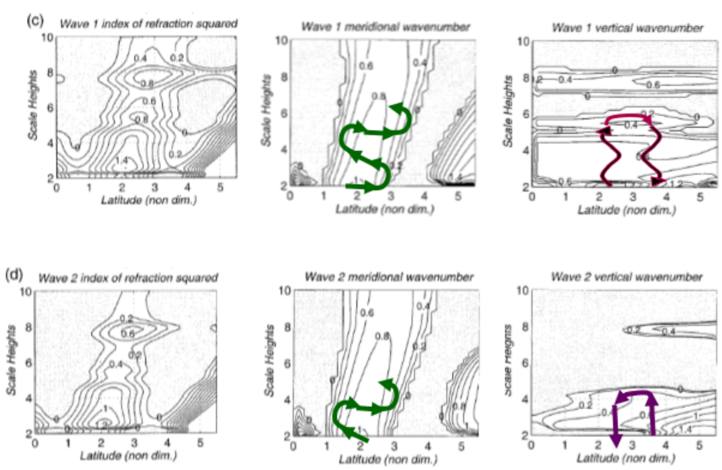
The basic state zonal wind (left) and meridional PV gradient (right, in units of β):



The corresponding wave geopotential height amplitude, for stationary zonal wavenumbers 1 (left) and 2(right), along with the latitudinal shape of the wave amplitude of the lower level forcing (the wave forcing was constant with latitude and time):



From these steady state wave solutions we calculate the meridional and vertical wavenumbers. In the following figure we show n^2_{ref}, m^2, and l^2 (respectively from left to right) for zonal wavenumbers 1 (top row) and 2 (bottom row), where wave evanescence regions are lightly shaded, and the wave propagation characteristics, including meridional and vertical reflection are schematically drawn:



We see that the n^2_{ref} gives a relatively good indication of the meridional propagation regions but less so of the vertical propagation regions. In particular, n^2_{ref} is positive at all heights in the meridional waveguide region, while m^2 is negative at upper levels. We also see that while n^2_{ref} suggests a diminished waveguide for wave 2 compared to wave 1, the wavenumbers show that the meridional waveguide is very similar and most of the difference is in the vertical propagation. For more details of this example see Harnik and Lindzen (2001).

Another example from observations that n^2_{ref} does not represent the wave geometry accurately enough is seen from the climatological n^2_{ref} vs m^2, and l^2 for Sep-Oct vs Jul-Aug using ERA40 from 1979-2001: while the index of refraction seems qualitatively similar, during Sep-Oct there is a vertically bounded meridional waveguide and downward wave reflection, while during Jul-Aug waves propagate vertically through the stratosphere (figures prepared by Tiffany Shaw):